THE DISTRIBUTION OF ELECTRIC POTENTIAL IN A CHANNEL WITH CONTINUOUS ELECTRODES FOR LARGE MAGNETIC REYNOLDS' NUMBERS

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It has been noted in the series [1, 3] that magnetic boundary layers are formed close to the walls when a conducting gas flows in a transverse magnetic field. If the exchange parameter [4] is small, then a dissipative layer arises on the electrode, increasing along the length of the electrode. It was shown in [1] that for the case of electrodes with weak longitudinal conductivity a boundary layer next to the anode is formed when there is a strongly marked Hall effect.

A channel with continuous electrodes was treated in [5], where the electric current distribution was determined using the method of matching asymptotic expansions [6].

This method is used below to find the electric potential distribution in a channel with continuous electrodes, while processes in the region next to the electrode connected with ionization and emission of electrons by the cathode are not considered. The linearized problem is solved in which the perturbation of flow by the boundary conditions is small.

1. We consider the motion of a nonviscous, nonthermally-conducting gas of high electrical conductivity in a flat channel which departs by only a small amount from a rectilinear channel of constant cross section. Let the external transverse magnetic field $H(0, 0, H_*)$ be uniform and constant. The distribution of gas-dynamic parameters in the channel is taken to be uniform for $x = 0$. The upper and lower walls of the channel $y = h + f(x)$, $y = f_1(x)$ for $x > 0$ are electrodes.

The distribution of parameters would remain uniform everywhere in a channel of constant cross section $f(x) \equiv f_1(x) \equiv 0$. In this case the induced emf is balanced by the potential difference created by the external source and applied to the electrodes.

If the cross section varies along the length of the channel, then electric currents flow in the channel and electromagnetic forces act on the gas. The parameter distribution becomes nonuniform.

Following [7], we carry out a linearization with respect to the small parameter ε ($f \sim \varepsilon h$) about the solution for a channel of constant cross section

$$u = 1, \quad v = 0, \ \rho = 1, \ T = 1, \quad H = 1, \ \varphi = -y \tag{1.1}$$

Here and in what follows the longitudinal and transverse velocity components u and v are given as ratios of the characteristic velocity u_*; the density and temperature as ratios of ρ_*, T_*; the magnetic field strength as a ratio of the constant applied field H_*; the electric potential as a ratio of u_*, H_*, h/c; and the coordinates x and y as ratios of the channel height h. (Parameter values at the entrance to the electrode section of the channel are taken as the characteristic quantities and are denoted by an asterisk.)

After linearizing quantities from the first approximation we have the following system of equations:

$$\frac{\partial u}{\partial x} = -\frac{1}{M^2}\frac{\partial \rho}{\partial x} - A^2\frac{\partial H}{\partial x}, \quad \frac{\partial^2 v}{\partial x^2} - \frac{1}{M^2-1}\frac{\partial^2 v}{\partial y^2} = -\frac{A^2M^2}{M^2-1}\frac{\partial^2 H}{\partial x \partial y}$$

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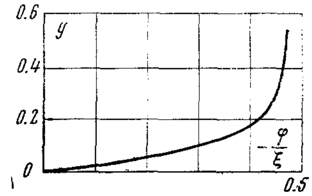

Fig. 1

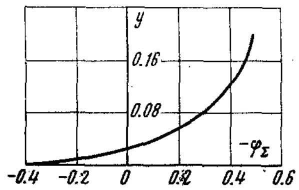

Fig. 2

$$\frac{\partial\rho}{\partial x}=\frac{A^2M^2}{M^2-1}\frac{\partial H}{\partial x}-\frac{M^2}{M^2-1}\frac{\partial v}{\partial y},\quad \frac{1}{R_m}\Delta H=\frac{M^2}{M^2-1}\frac{\partial v}{\partial y}+\frac{M^2(1-A^2)-1}{M^2-1}\frac{\partial H}{\partial x}$$

$$\Delta\varphi=\beta\left(\frac{M^2}{M^2-1}\frac{\partial v}{\partial y}-\frac{A^2M^2}{M^2-1}\frac{\partial H}{\partial x}\right)+R_m\left(\frac{\partial\varphi}{\partial x}-v\right) \tag{1.2}$$

$$\left(R_m=\frac{4\pi\sigma u_*h}{c^2},\quad A=\frac{H^*}{u_*\sqrt{4\pi\rho_*}},\quad M=\frac{u_*}{\sqrt{\gamma RT_*}}\right)$$

Here the Hall parameter for electrons β is determined from H_*; the conductivity σ and ratio of specific heats γ are constant; R is the gas constant; and c is the velocity of light.

This system, with the exception of the last equation, was treated in [5] for finding electric currents. The dimensionless parameters are the magnetic Reynolds' number and the Alfven and Mach numbers.

If R_m is large, then an expansion of v and H in powers of R_m^{-1} gives the following equations from the corresponding equations of system (1.2):

$$\frac{\partial^2 v^\circ}{\partial x^2}-\frac{1}{M_+^2-1}\frac{\partial^2 v^\circ}{\partial y^2}=0,\quad \frac{\partial v^\circ}{\partial y}+\left(1-\frac{1}{M^2}-A^2\right)\frac{\partial H^\circ}{\partial x}=0,\quad M_+=\frac{M}{\sqrt{1+A^2M^2}} \tag{1.3}$$

$$\frac{\partial^2 v_1}{\partial x^2}-\frac{1}{M_+^2-1}\frac{\partial^2 v_1}{\partial y^2}=-\frac{A^2M^2}{M^2-1-A^2M^2}\frac{\partial}{\partial y}\Delta H^\circ, \tag{1.4}$$

$$\frac{\partial v_1}{\partial y}+\left(1-\frac{1}{M^2}-A^2\right)\frac{\partial H_1}{\partial x}=\left(1-\frac{1}{M^2}\right)\Delta H^\circ$$

Here M_+ is the Mach number for the fast magnetoacoustic wave. In what follows only the case $M_+ > 1$ will be treated as in [5].

The first equation of (1.4) enables us to conclude qualitatively that a vertical velocity component ($v_1 \sim -J_x^\circ$, $J_x^\circ \sim \partial H^\circ/\partial y$) arises under the action of volume forces. A decrease of density at the anode and an increase at the cathode occurs in the case of negative J_x°. There is a downwash in the upward direction.

There are no volume forces in the zero-th approximation (1.3). A vertical velocity component arises as the result of velocity perturbation at the walls, which can come about as the result of a slight deformation of the walls, flowing in of fluid, electromagnetic forces, etc.

By way of an example, we consider a channel with an upper wall of the form given by the equation $y = 1 + f(x)$, and the lower wall by the equation $y = 0$.

The first equation of (1.3) has the following boundary conditions:

$$x=0,\quad v^\circ=0;\qquad y=0,\quad v^\circ=0;\qquad y=1,\quad v^\circ=f'(x)$$

The solution of (1.3) is given in [5]. It has the following form for the initial section of the channel:

$$v^\circ=\chi_+-\chi_-,\qquad H^\circ=-\frac{M_+^2}{\varkappa}(\chi_++\chi_-),\qquad \chi_{+,-}=\chi(x-\varkappa\pm\varkappa y)$$

$$\chi(x)=0,\quad x<0;\qquad \chi(x)=f'(x),\quad x>0;\qquad \varkappa=\sqrt{M_+^2-1} \tag{1.5}$$

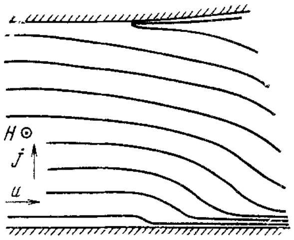

Fig. 3

The potential for $R_m \gg 1$ and finite β/R_m is obtained from (1.2) and (1.3)

$$\varphi^\circ = \int_0^x v^\circ dx + \frac{\beta}{R_m} H^\circ \tag{1.6}$$

2. The solution obtained H°, φ° does not satisfy the boundary conditions at the electrodes:

$$\frac{\partial H}{\partial y} - \beta \frac{\partial H}{\partial x} = 0, \qquad \varphi = \text{const} \tag{2.1}$$

To find a solution which would also fit close to the electrode, we use the method of matching asymptotic expansions [6]. According to this method, the external expansion obtained must be matched with the internal expansion.

The internal expansion is obtained with the help of the new variables

$$Y = y\sqrt{R_m}, \qquad V = \sqrt{R_m}\,[v - v^\circ(x)]$$

where v^0 is the velocity at the wall. [With this choice of variables $\partial V/\partial Y$ is of the order of unity, see (2.3).]

We have from (1.2)

$$\begin{gathered}\frac{1}{R_m}\frac{\partial^2 V}{\partial x^2} - \frac{1}{M^2-1}\frac{\partial^2 V}{\partial Y^2} = -\frac{A^2M^2}{M^2-1}\frac{\partial^2 H}{\partial x \partial Y}\\ \frac{1}{R_m}\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial Y^2} = \frac{M^2}{M^2-1}\frac{\partial V}{\partial Y} + \frac{M^2(1-A^2)-1}{M^2-1}\frac{\partial H}{\partial x}\\ \frac{1}{R_m}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial Y^2} = \frac{\beta}{R_m}\left(\frac{M^2}{M^2-1}\frac{\partial V}{\partial Y} - \frac{A^2M^2}{M^2-1}\frac{\partial H}{\partial x}\right) + \frac{\partial \varphi}{\partial x} - v^\circ(x) - \frac{V}{\sqrt{R_m}}\end{gathered} \tag{2.2}$$

Terms of the order $1/\sqrt{R_m}$, $1/R_m$ are rejected. The equation for the velocity is integrated with respect to Y:

$$\partial V/\partial Y = A^2M^2\,\partial H/\partial x + r(x) \tag{2.3}$$

Using the condition for matching with the external expansion for velocity, we find

$$r(x) = (1 - M^2)\,/\,M_+^2 \partial H^\circ\,/\,\partial x$$

It follows from (2.3) that the velocity change in the current layer is of the order $1/\sqrt{R_m}$. Thus, the velocity in the equation for the potential (2.2) turns out to be equal to the velocity at the wall.

Eliminating $\partial V/\partial Y$ from the induction and potential equations, we have

$$\frac{1}{1+A^2M^2}\frac{\partial^2 H}{\partial Y^2} = \frac{\partial H}{\partial x} - \frac{\partial H^\circ}{\partial x}, \qquad \frac{\partial^2 \varphi}{\partial Y^2} = \frac{\partial \varphi}{\partial x} + \frac{\beta}{R_m}\left[A^2M^2\frac{\partial H}{\partial x} - \frac{M^2}{M_+^2}\frac{\partial H^\circ}{\partial x}\right] \tag{2.4}$$

The boundary condition for the magnetic field in the case of a straight electrode is transformed as follows:

$$\frac{1}{\sqrt{R_m}}\frac{\partial H}{\partial Y} - \frac{\beta}{R_m}\frac{\partial H}{\partial x} = 0 \quad \text{or} \quad \frac{\partial H}{\partial x} = 0 \quad \text{for} \quad R_m \to \infty$$

We now formulate boundary conditions for the lower electrode ($v^0(x) \equiv 0$):

$$Y = 0, \quad \partial H/\partial x = 0, \quad \varphi = 0; \quad Y \to \infty, \quad H \to H^\circ(x), \quad \varphi \to \varphi^\circ(x) \tag{2.5}$$

The initial conditions are

$$x = 0, \qquad H = 0, \qquad \varphi = 0$$

3. The form of the wall $f(x)$ must be specified for concrete calculations. Let $f(x) = 1/2\ \varepsilon x^2$. We shall find the potential distribution in the region next to the lower electrode (anode) for the initial section of the channel $\varkappa \leq x \leq 2\varkappa$.

Equation (2.4) with conditions (2.5) is solved with the help of a Laplace integral transform [8].

$$\varphi = \frac{2\varepsilon M_+^2}{\varkappa} \frac{\beta}{R_m} x \left(-1 + (1+2z^2)\,\mathrm{Erf}(z) - \frac{2}{\sqrt{\pi}} z e^{-z^2}\right)$$
$$z = y / (2\delta \sqrt{x}), \quad \delta = 1 / \sqrt{R_m (1 + A^2 M^2)}, \quad \mathrm{Erf}(z) = 1 - \mathrm{erf}(z) \tag{3.1}$$

Here erf (z) is the error function and δ is the characteristic thickness of the current layer [5]. The coordinate origin is situated at the point $x = \varkappa$.

The variation of potential near the electrode is shown in Fig. 1 for $x = 0.5$, $\delta = 0.02$. It is clear from Fig. 1 that the potential varies sharply at a distance of order δ from the electrode.

The electric field strength E_y reaches a maximum value at the electrode

$$y = 0, \qquad E_y = \frac{2}{\sqrt{\pi}} \xi \frac{\sqrt{x}}{\delta}, \qquad \xi = 2\varepsilon\beta R_m^{-1} M_+^2 \varkappa^{-1} \tag{3.2}$$

The magnitude of the potential change in the current layer is proportional to the exchange parameter [4] ξ.

4. We consider the case when the channel walls are straight and a perturbation is introduced by changing the potential difference φ_Δ applied to the electrodes. When the solution of system (1.2) for v, H, φ is expanded in terms of $1/R_m$, the zero-th approximation gives $v^0 \equiv 0$, $H^0 \equiv 0$, $\varphi^0 \equiv 0$.

The solution $\varphi^0 \equiv 0$, which is the external expansion, does not satisfy the boundary condition $\varphi = \varphi_\Delta$ at the electrode. As in the preceding sections, a full solution is found by matching the external and internal expansions

$$\varphi = \varphi_\Delta \left[1 - \mathrm{erf}\left(\frac{y\sqrt{R_m}}{2\sqrt{x}}\right)\right] \tag{4.1}$$

Thus, the potential difference applied to the electrodes turns out to be concentrated in the region next to the electrode in accordance with the concepts of potential distribution in a channel given in [9]. The thickness of this region is of the order $1/\sqrt{R_m}$.

A solution is obtained in the form of a sum of (3.1) and (4.1) in the case of a curved wall and the presence of a potential perturbation at the electrodes. This can be seen if we use the system of solution of Secs. 1-3 with the appropriate boundary conditions for the potential.

The changes of potential next to the wall are illustrated in Fig. 2. Here φ_Σ is the sum of φ/φ_Δ and φ/ξ. The calculations were carried out for $\delta_1 = 1\sqrt{R_m} = 0.1$, $\delta = 0.02$, $\varphi_\Delta = \xi/2$.

It is clear from Fig. 2 that the potential variation close to the electrode is characterized by two dimensions δ_1 and δ, the thickness of the current layer.

From what has been explained above the potential distribution in the channel can be represented as shown in Fig. 3. Instead of weak discontinuities, the potential is assumed to change continuously in a thin layer of thickness of the order δ_1 [5].

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